# Cyclostationary Detection from Sub-Nyquist Samples for Cognitive Radios: Model Reconciliation

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***Abstract*—Cognitive Radio (CR) challenges spectrum sensing into dealing with wideband signals in an efficient and reliable way. CR receivers traditionally deal with signals with high Nyquist rates and low Signal to Noise Ratios (SNRs). On the one hand, sub-Nyquist sampling of such signals alleviates the burden both on the analog and the digital side. On the other hand, cyclostationary detection ensures better robustness to noise. Cyclostationary detection from sub-Nyquist samples has been considered via two main signal models that seem inherently different. In this paper, we show that those two models can lead to similar relations between the cyclic spectrum we wish to recover and the correlation between the sub-Nyquist samples. We show that we can then derive the minimal sampling rate allowing for perfect reconstruction of the signal's cyclic spectrum in a noise-free environment for both models in a unified way. We consider both sparse and non sparse signals as well as blind and non blind detection in the sparse case. Simulations show that our detector outperforms energy detection at low SNRs.**

## I. Introduction

The traditional task of spectrum sensing is now facing new challenges due, to a large extent, to Cognitive Radio (CR) applications. Today, CRs are perceived as a potential solution to spectrum overcrowdedness, bridging between its scarcity and its sparsity. Even though most of the spectrum is already owned and new users can hardly find free frequency bands, various studies [1] have shown that the spectrum is usually significantly underutilized. This motivates CR, which allows secondary users to opportunistically use the licensed spectrum when the corresponding primary user (PU) is not active [2]. One of the most crucial tasks in the CR cycle is spectrum sensing. CR requirements dictate new challenges for this task: sensing has to be performed in real-time, efficiently, with minimal software and hardware resources, and it has to be reliable and robust to noise.

In order to efficiently sample wideband signals, several new sampling methods have been proposed [3], [4] that reduce sampling rate in multiband settings below the Nyquist rate. These papers derive the minimal sampling rate allowing for perfect signal reconstruction in noise-free settings and provide sampling and recovery techniques. However, when the final goal is spectrum sensing and detection, reconstructing the original signal is unnecessary. In [5], the authors propose a method to estimate finite resolution approximations of the power spectrum. Power spectrum reconstruction is also considered in [6] both in the time and frequency domains based on energy detection. Unfortunately, the sensitivity of energy detection is amplified when performed on sub-Nyquist samples due to noise aliasing [7].

Another traditional detection technique is cyclostationary detection [8]. Processes, whose statistical characteristics vary periodically with time, are called cyclostationary [8]. A characteristic function of such processes, referred to as the cyclic spectrum or spectral correlation function (SCF), exhibits spectral peaks at certain frequency locations called cyclic frequencies. When determining the presence or the absence of a signal, cyclostationary detectors exploit one fundamental property of the SCF: stationary noise and interference exhibit no spectral correlation [8]. This renders such detectors highly robust to noise. In this paper, we propose to reconstruct the signal's SCF from sub-Nyquist samples and perform cyclostationarity detection, thereby obtaining an efficient, fast and frugal detector that is also reliable and robust to noise.

Several papers have considered cyclostationary detection from sub-Nyquist samples using two main signal models. In [9], [10], the authors consider a relation between the Nyquist and the sub-Nyquist SCF to retrieve the latter from the former. However, no analysis on the conditions for perfect SCF reconstruction is provided. Furthermore, this approach does not deal with the actual sampling scheme. In [11], [12], the sampling methods of [3], [4] are considered and the SCF of the analog signal is recovered from sub-Nyquist samples or from their correlation, respectively. In this paper, we aim at bridging between these two models and show that the same analysis of the minimal sampling rate for perfect SCF reconstruction in noise-free settings can be carried out for both.

We consider the class of purely wide-sense cyclostationary multiband signals, whose frequency support lies within several continuous intervals (bands) and study three different scenarii: (1) the signal is not assumed to be sparse, (2) the signal is assumed to be sparse and the carrier frequencies of the narrowband transmissions are assumed to be known, (3) the signal is sparse but we do not assume carrier knowledge. Our contribution is twofold. First, we derive the minimal sampling rate, allowing for perfect SCF reconstruction in a noise-free environment, for each one of the above three cases and for both signal models. We show that the rate required for spectrum reconstruction is a bit higher than half the rate that allows for perfect signal reconstruction, for each one of the scenarii, namely the Nyquist rate, the Landau rate [13] and twice the Landau rate [3], respectively. Second, we provide SCF recovery techniques that achieve these lower bounds. The performance of our detector in noisy settings is considered via simulations and shown to outperform energy detection.

This paper is organized as follows. In Section II, we present the cyclostationary multiband and multi-tone models. Section III describes the sub-Nyquist sampling stage and SCF reconstruction. In Section IV, we derive the minimal sampling rate for both models and for each one of the three scenarii described above. Numerical experiments are presented in Section V.

## II. System Models and Goal

### A. Analog Model: Multiband Model

Let $x(t)$ be a real-valued continuous-time signal, supported on $\mathcal{F} = [-1/2T_{\text{Nyq}}, +1/2T_{\text{Nyq}}]$ and composed of up to $N_{\text{sig}}$ uncorrelated purely cyclostationary transmissions, such that

$$x(t) = \sum_{i=1}^{N_{\text{sig}}} \rho_i s_i(t). \tag{1}$$

Here $\rho_i \in \{0, 1\}$ and $s_i(t)$ is a zero-mean purely cyclostationary with period $T_i, 1 \leq i \leq N_{\text{sig}}$, as defined below. The value of $\rho_i$

determines whether or not the $i$th transmission is active. The single-sided bandwidth of each transmission is assumed to not exceed $B$. Formally, the Fourier transform of $x(t)$ defined by

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \mathrm{d}t \tag{2}$$

is zero for every $f \notin \mathcal{F}$. We denote by $f_{\text{Nyq}} = 1/T_{\text{Nyq}}$ the Nyquist rate of $x(t)$. Let $\Gamma_x$ denote the support of $X(f)$.

A process $x(t)$ is said to be purely cyclostationary with period $T_0$ in the wide sense if its mean $\mathbb{E}[x(t)] = \mu_x(t)$ and autocorrelation $\mathbb{E}[x(t)x(t+\tau)] = R_x(t,\tau)$ are both periodic with period $T_0$ [8]:

$$\mu_x(t+T_0) = \mu_x(t), \qquad R_x(t+T_0,\tau) = R_x(t,\tau). \tag{3}$$

Given a wide-sense cyclostationary random process, its autocorrelation $R_x(t,\tau)$ can be expanded in a Fourier series

$$R_x(t,\tau) = \sum_{\alpha} R_x^{\alpha}(\tau) e^{j2\pi\alpha t}, \tag{4}$$

where $\alpha = m/T_0, m \in \mathbb{Z}$ and the Fourier coefficients, referred to as cyclic autocorrelation functions, are given by

$$R_x^{\alpha}(\tau) = \frac{1}{T} \int_{-\infty}^{\infty} R_x(t,\tau) e^{-j2\pi\alpha t} \mathrm{d}t. \tag{5}$$

The SCF is obtained by taking the Fourier transform of (5) with respect to $\tau$, namely

$$S_x^{\alpha}(f) = \int_{-\infty}^{\infty} R_x^{\alpha}(\tau) e^{-j2\pi f\tau} \mathrm{d}\tau, \tag{6}$$

where $\alpha$ is referred to as the cyclic frequency and $f$ is the angular frequency [8]. The following proposition relates the Fourier transform and the SCF of cyclostationary signals. [1]

**Proposition 1.** *Let $x(t)$ be a bandpass wide-sense cyclostationary process with period $T_0$. Then*

$$\mathbb{E}[X(\omega)X^*(\nu)] = 2\pi \sum_{m=-1}^{1} S_x^{m/T_0}(\omega)\delta\left(\omega - \nu + \frac{m}{T_0}\right), \tag{7}$$

*where $S_x^{m/T_0}(\omega)$ is defined in (6).*

From Proposition 1, the support of the SCF is dictated by that of the Fourier transform. Thus, we can recover $\Gamma_x$ by performing detection on the SCF.

### *B. Digital Model: Multi-tone Model*

Let $x(t)$ be a real-valued continuous-time signal defined over the interval $[0,T)$ and composed of up to $N_{\text{sig}}$ transmissions, such that

$$x(t) = \sum_{i=1}^{N_{\text{sig}}} \rho_i s_i(t), \qquad t \in [0,T). \tag{8}$$

Again, $\rho_i \in \{0,1\}$ and $s_i(t)$ is a wide-sense cyclostationary signal. Since $x(t)$ is defined over $[0,T)$, it has a Fourier series representation

$$x(t) = \sum_{k=-Q/2}^{Q/2} c[k] e^{j\frac{2\pi k}{T}t}, \qquad t \in [0,T), \tag{9}$$

where $Q/(2T)$ is the maximal possible frequency in $x(t)$. Each transmission $s_i(t)$ has a finite number of Fourier coefficients, up to $2K_{max} \leq Q+1$, so that

$$s_i(t) = \sum_{k\in\Omega_i} c[k] e^{j\frac{2\pi k}{T}t}, \qquad t \in [0,T), \tag{10}$$

[1]Due to lack of space, the proofs of the propositions are omitted here and will be detailed in a future paper.

where $\Omega_i$ is a set of integers with $|\Omega_i| \leq 2K_{\max}$ and $\max_{k\in\{\Omega_i\}} |k| \leq Q/2$. Thus, here the support $\Gamma_x$ of $x(t)$ is $\Gamma_x = \bigcup_{i=1}^{N_{\text{sig}}} \Omega_i$. We assume that each Fourier coefficient $c[k]$ of $x(t)$ is correlated with at most one other coefficient, namely $c[-k]$ since $x(t)$ is a real-valued signal.

For mathematical convenience, for this model we consider the Nyquist samples of $x(t)$, namely

$$x[n] = x(nT_{\text{Nyq}}), \qquad 0 \leq n < T/T_{\text{Nyq}}. \tag{11}$$

where $T_{\text{Nyq}} = T/(Q+1)$. Since $x(t)$ is wide-sense cyclostationary, $\mathbf{x} = \{x[n]\}_{n=0}^{Q}$ is wide-sense cyclostationary as well. Define $N = T/T_{\text{Nyq}} = Q+1$. From (9), the autocorrelation of $\mathbf{x}$, namely $r_{\mathbf{x}}[n,\nu] = \mathbb{E}\left[x[n]x^*[n-\nu]\right]$, has the following Fourier series expansion [8]

$$r_{\mathbf{x}}[n,\nu] = \sum_{\alpha=-Q}^{Q} r_{\mathbf{x}}^{\alpha}[\nu] e^{j2\pi\alpha(n-\nu/2)/N}, \qquad 0 \leq \nu \leq N-1, \tag{12}$$

where

$$\begin{aligned} r_{\mathbf{x}}^{\alpha}[\nu] &= \sum_{n=0}^{N-1} r_{\mathbf{x}}[n,\nu] e^{-j2\pi\alpha(n-\nu/2)/N} && (13)\\ &= \sum_{k,l,\,\text{s.t. } k-l=\alpha} \mathbb{E}\left[c[k]c^*[l]\right] e^{j\frac{\pi(k+l)}{N}\nu}, \quad -\frac{Q}{2} \leq k,l \leq \frac{Q}{2}. \end{aligned}$$

The Fourier series coefficients of $r_{\mathbf{x}}[n,\nu]$, namely $r_{\mathbf{x}}^{\alpha}[\nu]$, has the Fourier representation

$$\begin{aligned} s_{\mathbf{x}}^{\alpha}[f] &= \sum_{\nu=0}^{N-1} r_{\mathbf{x}}^{\alpha}[\nu] e^{-j\frac{2\pi f}{N}\nu} && (14)\\ &= \begin{cases} \sum_{k,\,\text{s.t. } 2k=\alpha} \mathbb{E}\left[c[k]c^*[-k]\right], & f=0 \\ 0, & \text{otherwise} \end{cases} && (15) \end{aligned}$$

Since each coefficient $c[k]$ is correlated with at most one other, $s_{\mathbf{x}}^{\alpha}[f]$ has at most $4N_{\text{sig}}K_{\max}$ non zero coefficients.

### *C. Problem Formulation*

We consider three different scenarii for each of the models.

*1) No sparsity assumption:* In the first scenario, we assume no *a priori* knowledge on the signal and we do not suppose that $x(t)$ is sparse, namely $2N_{\text{sig}}B$ can be of the order of $f_{\text{Nyq}}$ for the analog model, or $N_{\text{sig}}K_{\max}$ can be on the order of $Q+1$ for the digital one.

*2) Sparsity assumption and non blind detection:* Here, we assume that $x(t)$ is sparse, namely $2N_{\text{sig}}B \ll f_{\text{Nyq}}$ for the analog model, or $N_{\text{sig}}K_{\max} \ll Q+1$ for the digital one. We denote $K_f = 2N_{\text{sig}}$ and $K_f = 2N_{\text{sig}}K_{\max}$ for the first and second model, respectively. Moreover, the support of the potentially active transmissions, which corresponds to the frequency support of licensed users defined by the communication standard, is assumed to be known. However, since the PUs' activity can vary over time, we wish to develop a detection algorithm that is independent of a specific known signal's support.

*3) Sparsity assumption and blind detection:* In the last scenario, we assume that $x(t)$ is sparse but we do not assume any *a priori* knowledge on the signal's support.

In each one of the scenarii defined above, our goal is to assess which of the $N_{\text{sig}}$ transmissions are active from sub-Nyquist samples of $x(t)$. In order to determine which of the $N_{\text{sig}}$ transmissions are active, we first reconstruct the SCF of $x(t)$ for the analog model and of its Nyquist samples for the digital one. We then perform detection on the reconstructed SCF. For each one of the scenarii, we derive the minimal sampling rate enabling perfect SCF reconstruction in a noise-free environment. It can be shown [8] that white noise $w(t)$ exhibits no cyclic correlation, that is

$$S_w^\alpha(f) = 0, \quad \alpha \neq 0. \tag{16}$$

This property is the key for cyclostationary detection.

## III. Sub-Nyquist Sampling and SCF Reconstruction

### A. Analog Model

We begin with the analog model. For this model, we consider two different sampling schemes: multicoset sampling [3] and the MWC [4] which were previously proposed for sparse multiband signals. Since, both schemes lead to identical expressions of the signal's spectrum in terms of the samples, we obtain identical expressions for the signal's SCF as well.

*1) Sub-Nyquist Sampling:* Due to lack of space, we refer the reader to [3] and [4] for the multicoset and MWC derivations, respectively. We consider both sampling schemes to have $M$ channels, each sampling at the rate $f_s \geq B$. We report the relation between the known discrete-time Fourier transforms of the samples $z_i[n]$ and the unknown $X(f)$

$$\mathbf{z}(f) = \mathbf{A}\mathbf{x}(f), \qquad f \in \mathcal{F}_s. \tag{17}$$

Here, $\mathcal{F}_s = [-f_s/2, f_s/2]$, $\mathbf{z}(f)$ is a vector of length $M$ with $i$th element $\mathbf{z}_i(f) = Z_i(e^{j2\pi f T_s})$ and

$$\mathbf{x}_k(f) = X\left(f + K_k f_s\right), \quad 1 \leq k \leq N, \tag{18}$$

where $N$ is assumed to be even with $NT_{\text{Nyq}}$ being a multiple of the periods $T_i, 1 \leq i \leq T_{\text{Nyq}}$, and $K_k = k - \frac{N+2}{2}, 1 \leq k \leq N$. The deterministic matrix $\mathbf{A}$ is known and depends on the sampling parameters. We assume that those are chosen so that $\mathbf{A}$ is full spark in both cases [3], [4].

*2) SCF reconstruction:* We now derive a method for reconstructing the analog SCF for both sampling schemes. We will reconstruct $S_x^\alpha(f)$ from the correlation between $\mathbf{z}(f)$.

We define the autocorrelation matrix $\mathbf{R}_\mathbf{x}(f) = \mathbb{E}\left[\mathbf{x}(f)\mathbf{x}^H(f)\right]$ where $(.)^H$ denotes the Hermitian operation. From Proposition 1, the only non zero elements of $\mathbf{R}_\mathbf{x}$ are its diagonal elements $\mathbf{R}_\mathbf{x}(i,i) = S_x^0(f+(i-N/2-1)f_s)$ [5] and $\mathbf{R}_\mathbf{x}(i, N-i+1) = S_x^{(N-2i+1)f_s}(f)$, $f \in \mathcal{F}_s$, for $1 \leq i \leq N$. Clearly, our goal can be stated as recovery of $\mathbf{R}_\mathbf{x}(f)$, since once $\mathbf{R}_\mathbf{x}(f)$ is known, $S_x^\alpha(f)$ follows for all $f$.

We now relate $\mathbf{R}_\mathbf{x}(f)$ to the correlation of the sub-Nyquist samples. From (17), we have

$$\mathbf{R}_\mathbf{z}(f) = \mathbf{A}\mathbf{R}_\mathbf{x}(f)\mathbf{A}^H, \qquad f \in \mathcal{F}_s, \tag{19}$$

where $\mathbf{R}_\mathbf{z}(f) = \mathbb{E}[\mathbf{z}(f)\mathbf{z}^H(f)]$. It follows that

$$\mathbf{r}_\mathbf{z}(f) = (\bar{\mathbf{A}} \otimes \mathbf{A})\text{vec}(\mathbf{R}_\mathbf{x}(f)) = (\bar{\mathbf{A}} \otimes \mathbf{A})\mathbf{B}\mathbf{r}_\mathbf{x}(f) \triangleq \mathbf{\Phi}\mathbf{r}_\mathbf{x}(f), \tag{20}$$

where $\mathbf{\Phi} = (\bar{\mathbf{A}} \otimes \mathbf{A})\mathbf{B}$, and $\bar{\mathbf{A}}$ denotes the conjugate matrix of $\mathbf{A}$. Here $\otimes$ is the Kronecker product, $\mathbf{r}_\mathbf{z}(f) = \text{vec}(\mathbf{R}_\mathbf{z}(f))$, $\mathbf{r}_\mathbf{x}(f) = \text{vec}(\mathbf{R}_\mathbf{x}(f))$, and $\mathbf{B}$ is a $N^2 \times 2N$ selection matrix that has a 1 in the $j$th column and $[(j-1)N+j]$th row, and the $[j+N]$th column and $[j(N-1)+1]$th row, $1 \leq j \leq N$ and zeros elsewhere. Thus, by recovering $\mathbf{r}_\mathbf{x}(f), \forall f \in \mathcal{F}_s$, we recover the entire SCF of $x(t)$.

### B. Digital Model

In this model, we wish to recover $s_\mathbf{x}^\alpha[f]$ defined in (15). From (9),

$$\mathbf{c} = \mathbf{F}\mathbf{x}. \tag{21}$$

Here, $\mathbf{x}$ is given by (11), the entries of $\mathbf{c}$ are the Fourier coefficients of $\mathbf{x}$ (see (9)) and $\mathbf{F}$ is the $N \times N$ DFT matrix. Since $\mathbf{F}$ is invertible,

$$\mathbf{x} = \mathbf{F}^{-1}\mathbf{c}. \tag{22}$$

Define the autocorrelation matrix $\mathbf{R}_\mathbf{c} = \mathbb{E}\left[\mathbf{c}\mathbf{c}^H\right]$. From (15), the non zero elements of $\mathbf{R}_\mathbf{c}$ are those on the main and the second diagonals. Clearly, our goal can be stated as recovery of $\mathbf{R}_\mathbf{c}$, since once $\mathbf{R}_\mathbf{c}$ is known, $s_\mathbf{x}^\alpha$ follows.

We now relate $\mathbf{R}_\mathbf{c}$ to the correlation of sub-Nyquist samples. A variety of different sub-Nyquist schemes can be used to sample $x(t)$ [3], [4], [14]. Let $\mathbf{z} \in \mathbb{R}^\mathbf{M}$ denote the vector of sub-Nyquist samples of $x(t), 0 \leq t < T$, sampled at rate $f_s$ with $f_s < N/T$. For simplicity, we assume that $M = f_s T < N$ is an integer. We express the sub-Nyquist samples $\mathbf{z}$ in terms of the Nyquist samples $\mathbf{x}$ as

$$\mathbf{z} = \mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{F}^{-1}\mathbf{c} \triangleq \mathbf{G}\mathbf{c}, \tag{23}$$

where $\mathbf{A} \in \mathbb{R}^{M \times N}$ and $\mathbf{G} = \mathbf{A}\mathbf{F}^{-1}$. Since $\mathbf{F}$ is a DFT matrix, it is full spark. It follows that $\mathbf{G}$ is full spark as well, namely spark$(\mathbf{G}) = M+1$. Let $\mathbf{R}_\mathbf{z} = \mathbb{E}\left[\mathbf{z}\mathbf{z}^H\right]$ be the covariance matrix of the sub-Nyquist samples. We now relate $\mathbf{R}_\mathbf{z}$ to $\mathbf{R}_c$. From (23), we have

$$\mathbf{R}_\mathbf{z} = \mathbf{G}\mathbf{R}_\mathbf{c}\mathbf{G}^\mathbf{H}. \tag{24}$$

Vectorizing both sides of (24),

$$\mathbf{r}_\mathbf{z} = (\bar{\mathbf{G}} \otimes \mathbf{G})\text{vec}(\mathbf{R}_\mathbf{c}) = (\bar{\mathbf{G}} \otimes \mathbf{G})\mathbf{B}\mathbf{r}_\mathbf{c} \triangleq \mathbf{\Phi}\mathbf{r}_\mathbf{c}. \tag{25}$$

Here, $\mathbf{B}$ is as defined in the previous section, $\mathbf{\Phi} = (\bar{\mathbf{G}} \otimes \mathbf{G})\mathbf{B}$ is of size $M^2 \times 2N$ and $\mathbf{r}_\mathbf{c}$ is a vector of size $2N$ that contains the potentially non-zero elements of $\mathbf{R}_\mathbf{c}$.

We observe that we obtain a similar relation (20) and (25) in both models. Therefore, the next section refers to both.

## IV. Minimal Sampling Rate

### A. No sparsity Assumption

The following proposition provides conditions for the systems defined in (20) and (25) to have a unique solution.

**Proposition 2.** *Let $\mathbf{A}$ be a full spark $M \times N$ matrix with $N$ even ($M \leq N$) and $\mathbf{B}$ be a $N^2 \times 2N$ selection matrix that has a 1 in the $j$th column and $[(j-1)N+j]$th row, and the $[j+N]$th column and $[j(N-1)+1]$th row, $1 \leq j \leq N$ and zeros elsewhere. The matrix $\mathbf{C} = (\mathbf{A}^* \otimes \mathbf{A})\mathbf{B}$ is full column rank if $M^2 \geq 2N$ and $2M > N+1$.*

From Proposition 2, (20) and (25) have a unique solution even for $M < N$ which is our basic assumption. If $M \geq 4$, we have $M^2/2 \geq 2M-1$. Thus, in this case, the values of M for which we obtain a unique solution are $(N+1)/2 < M < N$. The minimal sampling rate is then

$$f_{(1)} = M f_s > \frac{N+1}{2} B = \frac{f_{\text{Nyq}} + B}{2}, \tag{26}$$

where $B \ll f_{\text{Nyq}}$. This means that even without any sparsity constraints on the signal, we can retrieve its SCF from sub-Nyquist samples by exploiting its cyclostationary property, whereas the measurement vector $\mathbf{z}$ exhibits no stationary nor cyclostationary properties in general. This was already observed in [10] for the digital model, without proof.

### B. Sparsity Assumption and Non-Blind Detection

We now consider the second scheme, where we have *a priori* knowledge on the frequency support of $x(t)$ and we assume that it is sparse. Instead of reconstructing the entire SCF, we exploit the knowledge of the signal's potential cyclic and angular frequencies in order to reconstruct only the potentially occupied bands. This will allow us to further reduce the sampling rate.

In this scenario, the only non zero elements of $\mathbf{R_x}$ are $K_f = 2N_{\text{sig}}$ diagonal elements and the corresponding $K_f$ second diagonal elements, where $K_f \ll N$. The reduced systems become

$$\mathbf{r_z} = \hat{\mathbf{\Phi}}\hat{\mathbf{r}}_\mathbf{x} \quad \text{or} \quad \mathbf{r_z} = \hat{\mathbf{\Phi}}\hat{\mathbf{r}}_\mathbf{c} \tag{27}$$

where $\hat{\mathbf{r}}_\mathbf{x}$ and $\hat{\mathbf{r}}_\mathbf{c}$ are vectors of size $2K_f$ that contain the potentially non zero elements of $\mathbf{r_x}$ and $\mathbf{r_c}$ respectively, and $\hat{\mathbf{\Phi}}$ contains the corresponding $2K_f$ columns of $\mathbf{\Phi}$.

The following proposition provides conditions for the systems defined in (27) to have a unique solution.

**Proposition 3.** *Let $\mathbf{A}$ be a full spark $M \times N$ matrix with $N$ even ($M \leq N$) and $\mathbf{B}$ be defined as in Proposition 2. Let $\mathbf{C} = (\mathbf{A}^* \otimes \mathbf{A})\mathbf{B}$ and $\mathbf{H}$ be the $2N \times 2K_f$ that selects $2K_f < 2N$ columns of $\mathbf{C}$ as defined above. The matrix $\mathbf{D} = \mathbf{CH}$ is full column rank if $M^2 \geq 2K_f$ and $2M > K_f + 1$.*

From Proposition 3, for $M \geq 4$, if $M > (K_f + 1)/2$, where $K_f \ll N$, we obtain a unique solution for (27). In this case, the minimal sampling rate is

$$f_{(2)} = Mf_s > \frac{2N_{\text{sig}} + 1}{2}B = (N_{\text{sig}} + 0.5)B. \tag{28}$$

Landau [13] developed a minimal rate requirement for perfect signal reconstruction in the non-blind setting, which corresponds to the actual band occupancy. Here, we find that the minimal sampling rate for perfect SCF recovery is equal to half the Landau rate plus half the maximal bandwidth of the narrowband transmissions.

### C. Sparsity Assumption and Blind Detection

We now consider the scheme where $x(t)$ is sparse, without any *a priori* knowledge on the support. In the previous section, we showed that $\hat{\mathbf{\Phi}}$ is full column rank, for any choice of $2K_f$ columns of $\mathbf{\Phi}$, provided $M^2 \geq 2K_f$ and $2M > K_f + 1$. Thus, for $M \geq 4$, we have spark$(\mathbf{\Phi}) = 2M - 1$. Therefore, if $\mathbf{r_x}$ or $\mathbf{r_c}$, is $(M-1)$-sparse, it is the unique sparsest solution of (20) or (25) respectively. In this case, the minimal sampling rate is

$$f_{(3)} = Mf_s > (2N_{\text{sig}} + 1)B, \tag{29}$$

which is twice the rate obtained in the previous case. As in signal recovery, the minimal rate for blind reconstruction is twice the minimal rate for non-blind recovery [3].

## V. Simulation Results

We now demonstrate the performance of the proposed detector in the presence of noise and compare it to energy detection. We consider the analog model and use the MWC [4] for the sampling stage.

In order to estimate the autocorrelation matrix $\mathbf{R_z}$, we first compute $\mathbf{z}(i), 1 \leq i \leq M$, using FFT on the samples $z_i[n]$ over a finite time window. We then estimate the elements of $\mathbf{R_z}$ using $P$ realizations of $z(\cdot,\cdot)$ as follows

$$\hat{\mathbf{R}}_\mathbf{z}(i,j,f) = \sum_{p=1}^{P} z_p(i,f)z_p^*(j,f), \quad f \in \mathcal{F}_s, \tag{30}$$

where $P$ is the number of frames for the averaging of the SCF.

We then perform cyclostationary detection on the reconstructed SCF. We use a single-cycle detector which computes the energy at several frequencies around $f = 0$ and at a single cyclic frequency $\alpha$. In the simulations, we consider AM modulated signals with cyclic features at $\alpha = 2f_c$, where $f_c$ is the carrier frequency of the signal to be detected.

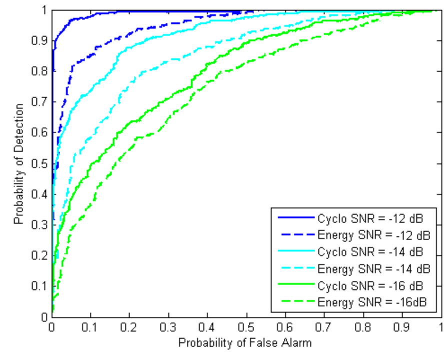


Fig. 1. Receiver operating characteristic (ROC) at SNR=-12dB, -14dB, for both energy detection and cyclostationary detection.

We consider a blind scenario where the carrier frequencies of the signals occupying the wideband channel are unknown and we have $N_{\text{sig}} = 3$ potentially active transmissions, with single-sided bandwidth $B = 150MHz$. The Nyquist rate of $x(t)$ is $f_{\text{Nyq}} = 10GHz$. We consider $N = 64$ spectral bands and $M = 7$ analog channels, each sampling at $f_s = 156MHz$. The overall sampling rate is $Mf_s = 1.09GHz$ which is 121% of the Landau rate and 10.9% of the Nyquist rate. The receiver operating characteristic (ROC) curve is shown in Fig. 1 for different SNR regimes (the averages were performed over $P = 15$ frames), illustrating that cyclostationary detection outperforms energy detection.

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